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Epistemology, didactics of mathematics and teaching practices

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Summary. *With this article we intend to give a contribution to a unitary picture of various terms and concepts, spread in the international community of those who deal with mathematics education, giving them back unity and looking for the historical roots of their introduction in that community. Although the different acceptation with which they appear today, many of these terms were introduced from the origin mainly by Guy Brousseau, striving for synthesis and ad hoc redefinition. They evolved in time and some these evolutions regard the most classical topics; we focus our attention on the example relative to the didactic contract.*

Resumen. *Con este artículo se quiere contribuir a dar una visión unitaria de varios términos y conceptos difusos en la comunidad internacional de quien se ocupa de didáctica de la matemática, restituyéndoles unitariedad y buscando las raíces históricas de su ingreso en dicha comunidad. Aún en sus diversas acepciones en las cuales hoy se usan, muchos de estos términos fueron introducidos desde sus orígenes gracias a la obra de Guy Brousseau, con un esfuerzo de síntesis y de redefinición ad hoc. Estos han evolucionado en el tiempo y algunas de dichas evoluciones atañen los temas clásicos; aquí nos limitamos al ejemplo relativo al contrato didáctico.*

Sunto. *Con questo articolo si intende dare un contributo ad una visione unitaria di vari termini e concetti oramai diffusi nella comunità internazionale di chi si occupa di didattica della matematica, restituendo loro unitarietà e cercando le radici storiche del loro inserimento in tale comunità. Pur nelle diverse accezioni con cui oggi compaiono, molti di questi termini furono introdotti fin dalle origini, principalmente ad opera di Guy Brousseau, con uno sforzo di sintesi e di ridefinizione ad hoc. Essi si sono evoluti nel tempo ed alcune di tali evoluzioni riguardano i temi più classici; qui ci si limita all'esempio relativo al contratto didattico.*

Resumo. *Com este artigo pretendemos fornecer uma contribuição para uma visão unitária de vários termos e conceitos já tão difundidos na comunidade internacional* *daqueles que trabalham com didática da matemática, restituindo-lhes unidade e procurando as raízes históricas de sua inserção nessa comunidade. Apesar das diferentes acepções com que aparecem hoje em dia, muitos desses termos foram introduzidos, desde sua origem, principalmente por Guy Brousseau, graças a um esforço de síntese e de redefinição ad hoc. Tais termos evoluíram no tempo e algumas dessas evoluções são relativas aos temas mais clássicos; aqui limitamo-nos ao exemplo relativo ao contrato didático.*

Résumé. *Cet article veut donner une contribution dans la direction d'une uniformisation des termes et des concepts très diffusés dans la communauté internationale de la didactique des mathématiques, en leur donnant ainsi unitarité et en même temps en recherchant leurs racines historiques de leur insertion dans cette communauté. Une bonne partie de ces termes ont été introduits, avec la même signification d'aujourd'hui, par Guy Brousseau, grâce à un effort de synthèse et de redéfinition ad hoc. Dans le temps, certains d'entre eux, concernant les thèmes les plus classiques, ont évolué; dans cet article on se borne à l'exemple relatif au contrat didactique.*

Zusammenfassung. *Dieser Artikel will sein Beitrag im Sinne der Standardisierung der Begriffe und der Konzepte geben, die in der internationalen Gemeinschaft der Didaktik der Mathematik sehr verbreitet sind. So gibt man ihnen Einheitlichkeit und gleichzeitig versucht man die historischen Wurzeln ihrer Einfügung in dieser Gemeinschaft. Viele dieser Begriffe waren mit der heutigen Bedeutung von Guy Brousseau eingeführt, dank einer seltsamen Anstrengung von Synthese und Neudefinierung. In der Zeit einige unter ihnen, die die klassischeren Themen betreffen, haben sich entwickelt; in diesem Artikel beschränkt man auf das Beispiel des didaktischen Vertrags.*

EPISTEMOLOGY, KNOWLEDGE AND BELIEFS

The term "epistemology" has been part of the didactics of mathematics since the 60's, bringing with it the different meanings that accompany it and that bring us to different "definitions" and interpretations in the various countries of the world and in multifaceted situations.

While I refer to Brousseau (2006a, b) for a comparative critical analysis of such terms and of their different circumstances, I inform you immediately that I will refer, even when I do not explicitly say it, to these two recent works of Brousseau and to many of his others, all of which are cited in the bibliography. Some of the sentences that follow are liberally taken from these texts, without changing their spirit. So as not to weigh down the text, I will not always explicitly cite the specific work of Brousseau to which I refer from time to time.

In our field of investigation:
an *epistemological conception* is a set of beliefs, of personal knowledge and of scientific institutional knowledge, which tend to tell what is the knowledge of individuals or of groups of people, their functioning, the way of establishing their validity, of acquiring them and therefore, of teaching and learning them;
epistemology is an attempt to identify and unify different epistemological conceptions relative to determined sciences, to movements of thought, to groups of people, to institutions or to cultures.

For some of these terms, we follow the definitions given in D'Amore, Fandiño Pinilla (2004):

- *belief* (or credence): opinion, set of judgments and of expectations, that which one thinks with regards to something;
- the set of someone's beliefs (A) about something (T) gives the *conception* (K) of A relative to T; if A belongs to a social group (S) and shares with the others who belong to S that set of beliefs relative to T, then K is the conception of S relative to T. Often, in place of "conception of A relative to T" one speaks of "the image that A has of T".

For other terms, we draw from encyclopaedias or reliable manuals:
for *cultural knowledge* we mean a set of things known or reproducible attitudes, acquired through study or experience.
In the sphere of cognitive psychology, *cultural knowledge* is distinguished from *personal knowledge*[1]
cultural knowledge are data, concepts, procedures or methods which exist outside of each subject which knows and that are generally codified in reference works, manuals, encyclopaedias, and dictionaries;
personal knowledge are inseparable from the knowing subject; that is, a-personal knowledge does not exist; a person who internalises a cultural knowledge *taking awareness of it*, transforms it into personal knowledge.

Now we turn to the didactic discourse; it is broad and can have to do with origins from various roots, one of which is found in the debate between

Didactics and pedagogy

The *great didactics* of Comenius has been a diehard: «a single method is enough to teach all subjects….the arts, sciences and languages» (Comenius, 1657).

[1] In literature we do not find this distinction and the term "knowledge" is generally used with both meanings. We have, throughout the whole paper, translated the corresponding Italian plural terms "saperi" and "conoscenze" with "knowledge", which is improper, to maintain the sense of generality and abstraction conveyed by the Italian words.

It has taken centuries to arrive at establishing, in a definitive way, that didactics can be, are, specific. This has helped (general) didactics to free itself from the yoke of pedagogy and the specific (disciplinary) didactics to achieve a status of its own.[2]

Analogously to the direction that just above we desired to dedicate to epistemology, we can say that the didactics of a personal knowledge (of an object, a fact, a discipline…) can then be redefined as a social project to help this knowledge be acquired by means of an organism.

It is in these "social" conditions that it is important to highlight some possible peculiarities of the

Didactics of mathematics

1. The didactics of mathematics (which we consider as one aspect of the more general mathematics education) is the art of conceiving and conducting conditions that can determine the learning of a piece of mathematical knowledge on the part of a subject (who can be any organism involved in such activity: a person, an institution, a system even an animal).[3] Learning is here understood as a set of modifications of behaviours (thus of performance) which signal, to a predetermined observer, the second subject in play, that this first subject has some personal knowledge (or a competence)[4] or a set of personal knowledge (or competences), which involve the management of different representations, the creation of specific beliefs, the use of different languages, control of a set of repertories of suitable references, of proofs, of justifications or of obligations. These conditions must be able to be placed into operation and be intentionally reproduced. In this case, one speaks of didactic practices.[5]

2. These didactic practices are themselves the same "conditions" and therefore in turn objects of study. Didactics presents itself then as the study of such conditions, under the form of projects and of real achievements.

3. The scientific studies – of the experimental kind – in this field necessitate the explication of concepts and methods which must be submitted to verification of their coherence and appropriateness to the specific contingency. Certain theories, as for example the theory of didactic situations, have as their object saying what didactics study.

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[2] Nevertheless, one could still obtain much from this work, not yet completely explored by the modern critics. I promise myself, once again, to do it.

[3] In this text, *art* is understood as the translation of the Latin *ars*, that is, a set that is not easily distinguishable between the current terms *art* and *artisan*; *artist* was, in the Latin understanding, any artist (in the modern sense of the term), but also any artisan. Better still, in the Latin world these two figures flowed together into one and it was no possibility of distinction.

[4] On the distinction between knowledge and competence, please see (D'Amore, Godino, Arrigo, Fandiño Pinilla, 2003).

[5] On the theme of practices, please see (D'Amore, 2005; D'Amore, Godino, 2006).

Amongst the subjects of didactic study, an absolutely fundamental role, but sometimes unexpressed is up to

Milieu (environment, means)

From the theory of situations, we know that the teacher must bring out in the student behaviours that the student himself, to show his knowledge, should take on autonomously. It seems to be a paradox. Better yet, it is a paradox. The only solution is to involve a third element, the *milieu*, and to do so in such a way that the pupil's response refers exclusively to the necessity of the *milieu*, which the teacher knows well, or has been predisposed by him as the aim. The art of the teacher is then in the organisation of a relationship between the pupil and *milieu*, which:

- on one hand, leaves a reasonable uncertainty which the subject's knowledge must reduce;
- on the other hand, it sees that such a reduction can truly happen, that is, with a limited degree of uncertainty, from the teacher's point of view.

From here one understands the role of *milieu*, fundamental for understanding the functioning of the

Theory of mathematical situations

The *theory of mathematical situations* (a-didactic situations) has as its goal defining the conditions in which a subject is lead to "do" mathematics, to use it or invent it without the influence of determined and explicit specific didactic conditions from the teacher.

It aims at the creation, organisation, and use of problems that lead to the construction of mathematical concepts and theories on the part of a subject with some properties and a minimum of personal knowledge; such to render the carrying out of the process, determined by the situation, quite probable.

On the basis of the last two points, the situations can be thought of as systems of interaction of one or more subjects with a *milieu*, subjects which necessitate previous personal knowledge to be able to act.

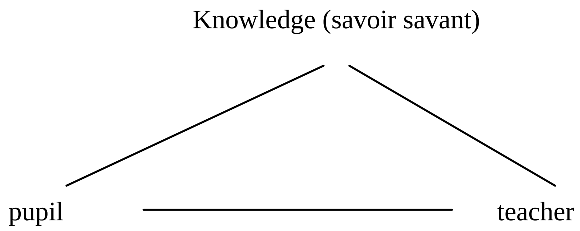
The elements of the theory are defined on the basis of their function in a situation. This is analogous to the method generally used in mathematics, according to which an object is defined on the basis of relationships with other objects (axioms and definitions).

Thus, a *didactic event* becomes a set of facts interpretable by the evolution of a didactic situation. Such an interpretation is one of the aims of the didactics of mathematics. It brings us to the idea of *microdidactics* understood as the study of the conditions of the diffusion or exchange of personal knowledge (for example by means of lessons), between people, social, economic or cultural organisations.

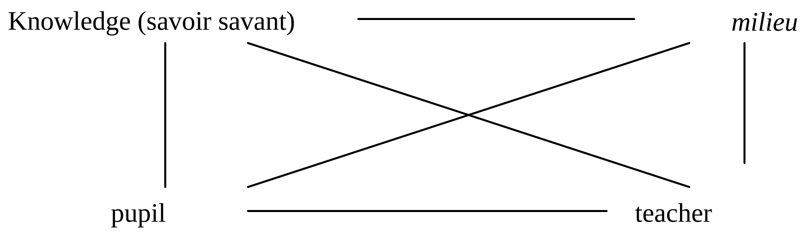
To schematically represent this situation, in recent history, we have turned to various schemes that Brousseau calls

"Polygons" of didactics

The most famous and cited is the triangle of didactics:[6]

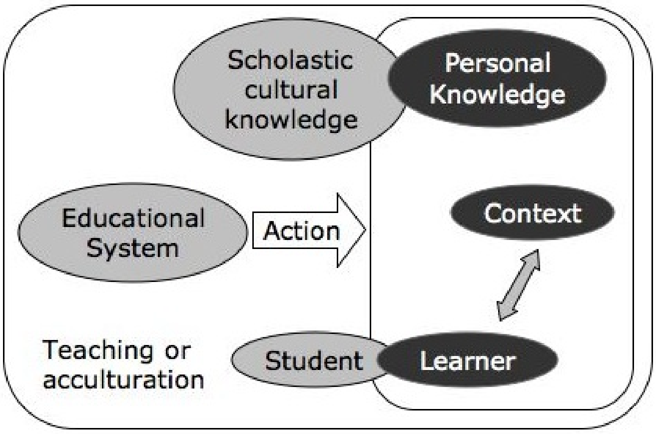


However in such a diagram milieu does not appear and this reveals its insufficiency. Introducing this new "vertex", one can pass to a quadrilateral of didactics:



Also this diagram reveals its insufficiency, as soon as one considers that it does not highlight the difference between the scholastic "cultural knowledge", to be taught or taught, and the pupil's "personal knowledge", which does not coincide amongst themselves and that function according to different modalities. Moreover, also the peculiarities of the activities of the subject who learns are different, which brings us at least a "hexagon of didactics", rendered by Guy Brousseau in this diagram which highlights its functional significance.

[6] A critical and constructive analysis of the triangle of didactics can be found in (D'Amore, Fandiño Pinilla, 2002).



In the future, we will always have to enter more profoundly into the analysis of this diagram and its implicit relational meanings, and use it for studying the didactic events in the classroom.

Before passing to the meaning of “result of research in the didactics of mathematics” and, finally, to examples of the didactic contract, I want to underline, in specific and exemplary cases, how the relationships between didactics and epistemology are revealed only during the carrying out of research.

EPISTEMOLOGICAL OBSTRACLES: A HISTORICAL EXAMPLE WHICH CHANGED THE FACE OF DIDACTICS

It is well known that Guy Brousseau studied, for almost three decades (from the early 60’s through the 80’s) how one learns natural numbers and their structure. Throughout the 60’s (and in some cases, even after) certain ideas dominated which today are found to be curious; based on different “theories” on the learning of natural numbers on the part of children at the beginning of primary school. For example, it was believed obvious that there was a learning necessity of proceeding in the oral and written learning of natural numbers according to the articulation of the ordinal succession, first 1, then 2, then 3, and so on. There was heavy insistence on the use of pre-built materials based on this supposed necessity and which therefore reinforced it.

I think it is well known that Brousseau fully *demonstrated* how this was absolutely false and how the learning of natural numbers comes “in leaps”. I think his anthropological and epistemological study on the writing of numbers, is well known, comparing three different systems: (1) the so-called one “of Robinson Crusoe” (one notch for each unit, with a space $\rightarrow$ between the single notches), (2) the one of the ancient Romans, and (3) the one of certain structured materials pre-packaged for the purpose, with the current indian-arab base ten positional system.

He introduced the idea of "zone of best effectiveness" to show how numerical intervals exist in which one system of writing is more effective than another. For example, for the interval 1-3, Robinson's method is more effective than Roman writing and than our decimal numbering (both for its use and for learning). On the interval 100-1000, the order is reversed. Continuing along this road and studying other intermediary intervals, Brousseau arrived at suggesting "learning by leaps" which he proposed at the end of 1965 in a book intended for the primary school and published by Dunod (Brousseau, 1965). Such learning can come "by invention", as is typical of a-didactic situations.

The study continues with the learning of operations, but the method could expand to the study of the learning of an algorithm and to that of a mathematical theory. And, from here, to that of each personal knowledge…

The leaps of "informational" complexity are therefore more frequent and better justified in mathematical discovery, compared to step by step progression. On the other hand, pupils encounter a lot of difficulties in the zone of transition between certain numeric intervals. These two symptoms lead Brousseau to the hypothesis according to which the phenomenon of the leaps was general, at least in mathematics, and that its analysis had to be the basis of every didactic engineering.

This idea was first presented in 1976: thirty years ago!

It was these types of studies which lead, in opposition to what Gaston Bachelard (1938) declared with regards to the non-existence of epistemological obstacles in mathematics, to the mooring instead of this concept in didactic research. The comprehension of natural numbers demands, for example, a certain way of conceiving of these numbers and their operations: a natural number like 4 has a successive one; its product by another natural number will be greater than itself etc. Some of these properties are wrong when 4 becomes a rational number. For example, it no longer has the successive one. However, the student does not come to this passage and continues to "force" the properties of N also in Q. It is for this that students are found who assert, in Q, that 2.33 is successive to 2.32, helped in this even by some textbooks. Besides, for example, $0.7 \times 0.8 = 0.56$ is smaller than each of the factors, a disconcerting novelty which places in crisis the previously acquired personal knowledge.

The student, I was saying, almost does not realise this transformation of cultural knowledge. The teacher calls "multiplication" or "division" the new operations that he would like the pupils to "recognise" and assimilate to the preceding ones. The personal knowledge of natural numbers is indispensable for acquiring that of the rational numbers, but at the same time, it is an obstacle to this acquisition. This phenomenon creates misunderstandings and serious, invisible difficulties since the obstacle is hidden

inside of a cultural knowledge that works, but which is “local” and not generalizable to the mathematical object which should be learned.
This is the same sense of the idea of *epistemological obstacle*.
Now there remains to clarify well what is intended therefore by

“Results” of researches in the didactics of mathematics

Those things that we call “results” are, according to Brousseau, principally of two typologies:

- assertions (not contradicted) about quite a vast a field of experiences;
- refusal of beliefs contradicted by experiences.

Examples of results of the first type.

1. The personal knowledge that a subject can have of a particular mathematical cultural knowledge depends on the circumstances in which he has the occasion to use it. This is an axiom that grounds the theory of didactic situations which has never been contradicted.

2. It is possible to teach mathematics in a relatively direct way with a correct implicit meaning, thus limiting the didactic transposition.

3. It is possible to establish reasonable reproducible conditions for the *use* and the *acquisition* of mathematical personal knowledge in the form of systems (the “situations”). It is also relatively possible to establish (different) reasonably reproducible conditions for their *teaching*.

4. It is possible to communicate these conditions to the teachers. It is preferable, from many points of view, to communicate to them the situations rather than closed algorithms or too general indications. This last point has various social repercussions.

Examples of results of the second type.

1. The idea that the individual history of a subject who learns is expressible in terms of added succession of definitive personal knowledge, from infancy to university, is a coarse approximation. Taken literally, it generates misunderstanding, fallacious decisions, and failures. The conceptions turn out to be limited and deformed, often in a hidden way. Some retakes and reorganisation of mathematical cultural knowledge are absolutely necessary, even when such knowledge seems acquired.

2. Radical constructivism is an appropriate theory for a-didactic situations, but inappropriate for didactic situations. The institutionalisation of personal knowledge is an indispensable stage of learning; it is constituent of cultural knowledge in relationship to personal knowledge.

3. The current descriptions of pupils' mathematical personal knowledge (in an administrative and popular sense) are inappropriate. These descriptions lead parents, teachers, and administrators to underestimate the results of didactic activity. The use of these descriptions for taking decisions about the policy of teaching, curriculum, laws, and bodies without adequate didactic awareness leads to disastrous consequences. Moreover, it leads the teachers to centre on the acquisition of cultural knowledge on the part of the pupil, ignoring the problem of maintaining personal knowledge which is indispensable for the genesis of cultural knowledge themselves. This degeneration of the didactic environment in the end causes an effective lowering of the pupils' personal and cultural knowledge, which re-feeds the system of negative decisions.

From all of this emerges the necessity of knowing uses and necessities of the epistemological knowledge on the part of the teacher. Still, there exists an epistemology (Speranza, 1997; Brousseau, 2006a) which can be called

The spontaneous epistemology of the teachers

To take their decisions in the classroom, teachers explicitly or implicitly use every kind of personal knowledge, method, and belief about ways of finding, learning, or organising cultural knowledge. This epistemological baggage is essentially constructed empirically for responding to didactic necessities. This, sometimes, is the only means that allows them to propose the chosen didactic processes and to have them accepted by their pupils and by their environment. The set of beliefs of the teachers, of the pupils, or of the parents about that which is better for teaching, for learning, and for understanding cultural knowledge in play, constitutes a practical *epistemology* which is impossible to ignore and eliminate. Philosophic or scientific epistemology is far from being able to claim taking on this role.

Spontaneous epistemology establishes its roots in an ancient practice, given that the tendency to communicate experiences from one generation to the successive one is an essential characteristic of humanity. It would be absurd to oppose it against the scientific knowledge. It is necessary to respect it, understand it, and study it experimentally, as a "natural" phenomenon.

The usefulness of the introduction of epistemology and of scientific theories attached to the training of teachers is thus presented from a new point of view (D'Amore, 2004).

However, before moving on, it is necessary to give a precise example of the working of the two kinds of epistemology that we have presented. We will do this, by means of an example gathered from Brousseau (2006b).

THE DOUBLE CONSTRAINT OF DIDACTIC SITUATIONS

The teacher proposes a problem to his pupils which he maintains is *analogous* to a problem that he had previously proposed to them, but which they had failed to solve. He hopes that they will recognise the similarity and that they will use the correction and explanations which he gave to *reproduce* the same solution method, in such a way as to confront the new situation with success. Therefore, he strongly recommends to his students to look for and use this analogy. This procedure is successful, that is, has success in the teacher's eyes. However, in reality it constitutes an epistemological fraud. The pupil produces an exact response, but not because he has understood its mathematical or logic necessity beginning from the statement, not because he has "understood and solved the problem", not because he has learned a mathematical object, but simply because he has established a similarity with another exercise. He has not done anything other than reproduce a solution already done by others for him. What is worse, he is aware that this was the request of the teacher. He will think he has understood the mathematical question in play, while he has not done anything other than interpret a didactic intention explicitly expressed by the teacher and supply the expected answer.

This "abuse of the analogy" which Guy Brousseau has highlighted since the 70's, but on which many didactic actions in the classroom are still based, is one of the most common forms of what he himself defined "Jourdain effect"; one of the effects of the didactic contract. The teacher obtains the expected answer with means that have no value and make the pupil (the family, the institution) think he has carried out a mathematical activity which was the aim to be reached.

The activity of the pupil therefore must respond to two incompatible restraints:

- the one determined by the a-didactic conditions which lead to an original answer and the organisation of specific personal knowledge;
- the one determined by didactic conditions which have as their aim the production of the expected answer, independent of its modality of production.

This example demonstrates that if epistemology and the cognitive sciences can study and prove right, some of the pupils' answers under the first restraint alone, they can not pretend to help the teachers ignoring the second. The didactic restraints will end up oppressing the cognitive restraints. They transform the nature itself of personal knowledge and its functioning. The teaching thus becomes a simulation of the genesis of personal knowledge.

All this explains the reason for the necessity of specific studies, of the didactics of mathematics, which can not trace back either to theories of learning or to exclusively epistemological studies. The didactic contract, for its strength and its extraordinary characteristics, will be the subject of the following examples. Guy Brousseau revealed importance of it to the scientific community in the 60's.

THE INTERPRETATION OF CLASSROOM EVENTS IN THE LIGHT OF DIDACTIC RESEARCH TOOLS: THE EXAMPLE OF THE DIDACTIC CONTRACT

From a research project on problems with missing data and on the attitudes of the pupils facing a problem of this type (D'Amore, Sandri, 1998) here is a text proposed in the 3rd year of primary school (pupils 8-9 years old) and in the 2nd year of middle school (pupils 12-13 years old):

«Giovanna e Paola go to do the shopping. Giovanna spends 10.000 lira and Paola spends 20.000 lira. At the end, who has more money in her purse, Giovanna or Paola?».

Here is a prototype of the most diffuse kind of answer in the 3rd year of primary school. I choose the protocol of Stefania's answer, which I report exactly as it was written by the pupil:

> Stefania:
>
> In the purse remains more money giovanna
> 30-10=20
> 10×10=100

In dealing with a "contract", for a long time I've been tracking down some "constants of behaviour" which can be called "clauses";[7] in the case in question , two of these play a formidable role:
clause of the expectations: the teacher certainly expects an answer, therefore I must supply it, the sense of the text does not matter;
clause of the constancy: the teacher has *always* given problems with a written text with words and some numbers and, to produce the result, I have *always* worked on these numbers with some operations; if it is *alwa*ys like this, it will certainly have to be the same *also* this time.

The "Giovanna" answer (58,4% of such answers in the 3rd year of primary school, age of the pupils from 8-9) is justified by the fact that the student maintains that, if the teacher gives him a problem to solve, *it must be solvable*. Therefore, even if he realises that the piece of data of the initial sum is missing, he implicitly invents it, more or less as follows: «This problem *must* be solved. Therefore, perhaps Giovanna and Paola started out with the same sum». At this point, *the answer is correct*; Giovanna spends less and therefore she has more left. And, this justifies the written part of Stefania's answer. After that, another mechanism tied to another clause activates (of the sort: image of mathematics, presupposed expectations on the part of the teacher): «This can't be enough. In mathematics one has to do calculations, the teacher certainly expects them». At that point, the critical check collapses and…any calculation is fine.

[7] For this idea, that I began to use in the early 90's, I made use of Chevallard (1988) who, speaking of metacontract, cited this term even if in another sense.

In the work D'Amore, Sandri (1998), we called this clause of the didactic contract "the need for formal justification" (nfj), studying it in every detail (also in successive works). Such an nfj clause is very present in the middle school as well (age of the pupils:11-14). [The percentage of "Giovanna" answers goes down from 58,4% in the 3rd year of primary school (8-9 years old) to 24,4% in the 2nd year of middle school (12-13 years old); but only 63,5% of the pupils of the 2nd year of middle school reported, in some way, the impossibility of giving an answer; therefore 36,7% gave an answer; over a 1/3 on average].

Here is the prototype of an answer received for the same problem proposed in the 2nd year of middle school. I chose the answer protocol of one pupil, reporting it exactly as she produced it:

> Silvia:
>
> In my opinion, the one with more money in her purse is Giovanna
> because:
> Giovanna spends 10.000 while Paola spends 20.000
> 10.000 20.00 [sic]
> Giovanna Paola
> 20.000-10.000=10.000 (Giovanna's money)
> 10.000+10.000=20.000 (Paola's money)

In Silvia's protocol one recognises the same clauses of the didactic contract in action as placed in operation in Stefania's protocol, but its analysis is more complex. For the first thing, one notes a more demanding attempt at logical and formal organisation. Silvia, then, at first spontaneously writes "Giovanna" without doing any calculations, because she reasoned like Stefania. Then, however, because of the clause nfj, she maintains that she *must* produce calculations. It is probable that she realises, even in a confused way, that the operations she is doing are not tied to the logic of the problem; she is only doing them because she maintains that she *must do* some calculations. However, as absurd as it is, she ends up using them as if they were plausible. As a matter of fact, given that from these senseless calculations she obtains a result which contrasts with that given intuitively, she prefers to violate her own intuition and rather accept that which she obtained formally; the calculations give "Paola" as the answer and not "Giovanna" as instead she had supposed, and therefore she crosses out "Giovanna" and in her place writes "Paola":

> In my opinion, the one with more money is ~~Giovanna~~ Paola
> because:
> Giovanna spends 10.000 while Paola spends 20.000
> 10.000 20.00
> Giovanna Paola
> 20.000-10.000=10.000 (Giovanna's money)
> 10.000+10.000=20.000 (Paola's money)

The didactic contract, which this time is dictated by a formal image, (empty and deleterious) of mathematics, has won, defeating reason…

In D'Amore (1993), I tell of an experience based on the following text, given in a primary school to different classes:

The 18 pupils of the 2nd year want to do a day trip from Bologna to Verona. They must take into account the following data:
- two of them cannot pay;
- from Bologna to Verona it is 120 km;
- a small coach with 20 places costs 200.000 lira a day plus 500 lire per kilometre (including the cost of the motorway).

How much will each pupil spend?

It is pointless to say that this has to do with a complex problem, that they really wanted to carry out the planning for a trip, that the students should have discussed the problem and looked for the solution as a group etc.

In fact, the great majority of the students, confronted with the solution of this problem, commit a recurring error; they do not take into account the return trip and therefore calculate the total expense with an erroneous expression: 500×120 + 200000 in place of (500×120)×2+200000.

There is a vast bibliography on points of this type which tends to justify these choices. One of the most recurring justifications is a sort of … strategic or sentimental forgetfulness; the outward journey of a trip is an emotively strong time, the return no.

Seeking to better understand the question, I divided the problem into various components or phases, with many specific partial "questions", but the error repeated itself. I then suggested to some teachers to have the students mime the scene of the outward journey and return, to draw the various steps of the trip. One interesting case which I encountered and which I described in D'Amore (1993) is that of a child who drew the following chart:

Bologna → Verona
120 Km

Verona → Bologna
120 Km

Therefore, there is perfect awareness of the fact that in a trip there is an outward journey and return, but then the same child, when he had to solve the problem, again used only the information for the outward journey.

One of the justifications most presented by the children in the interviews was that they didn't feel authorised to use a piece of data that did not explicitly appear in the text. The *sense of the request* counts for little in mathematical problems. That which counts is to make use of the numeric data given expressly as such. One of the children interviewed said, "If you wanted us to also calculate the return, you should have said so." And the gap that the student sensed is evident; in not one of the data given does it appear legitimate to double the expense for the kilometres travelled. The didactic contract imposes rules of behaviours and, as Brousseau explained, the didactic constraints impose themselves with respect to the a-didactic ones.

It turns out to be very interesting to read the approach of the students when confronting the following famous problem of Alan Schoenfled (1987):

An army bus transports 36 soldiers. If 1128 soldiers have to be transported by bus to the training camp, how many buses will need to be used?

Of the 45,000 15 year old pupils tested in the USA by Schoenfeld, only less than a quarter (23%) managed to give the expected answer that is 32. The American researcher stated therefore that very few students are able to reread the sense of the question, daring to write 32, in fact not obtained formally in the operation, and he proposed as a cause of this behaviour questions relative to metacognitive facts. The explanation of this event, according to the Author is in a gap in the metacognitive processes, therefore in the fact that the students, once having obtained the numeric result after an arithmetic process for the solution of the problem, are not able to go back over their own steps, reread the text critically, realise the actual request, and interpret the result obtained to give a correct answer.

After several years, we have recently decided to reanalyse the same situation (D'Amore, Martini, 1997), interviewing the pupils, something which Shoenfeld did not do, and we found some new things. The test was done at several scholastic levels adding a variable, that is, leaving the students free to use, or not, a calculator. We had many answers of the type 31.333333 overall on the part of those who used the calculator; other answers: $31.\overline{3}$ and 31.3.

The semantic control, when there is one, leads some to write 31 (the busses «can not be broken up»), but very few felt *authorised* to write 32. Amongst those who used the calculator, then, there were 0% of answers of "32".

The interview demonstrates that the student does not feel authorised to write that which does not appear. Even if he does a semantic check of the bus as an object that is not divisible into parts, that does not authorise him to write 32. There is even he who does not feel authorised to write 31. One cannot simply speak of 'error' on the part of the student, at least if one does not mean, by 'error', the inability to check, once having obtained the answer, if it is semantically consistent with the question asked. However, another mechanism then goes off; the student is not ready to admit that he has made an error and prefers to speak of "trick" or "trap". For the student, a mathematical error or in mathematics is an error of calculation or assimilable to it. He does not accept that a mistaken semantic interpretation is considered an error.

A long and systematic study of this test, also carried out through numerous interviews of the students, revealed that "the guilty party" of this behaviour is a clause of the didactic contract to which we have given the name 'the clause of the formal proxy'. The students reads the text, decides on the operation to carry out and the numbers with which he has to work. At that point, indeed, the clause of the *formal proxy* is set off. It is no longer for the student to reason and check, he no longer considers what follows as his personal responsibility. Both when he does the calculations by hand and *even more* if he uses a calculator, that clause starts which disengages his rational and critical faculties of control. The involvement of the student is finished and now it's up to the algorithm or better yet to the machine, to work for him. The student's next task will be that of transcribing the result, whatever it is and it does not matter what it means in the problematic context he started with.

This fact also explains another didactic event. Efraim Fischbein's example (1985) is well known:

P1. A bottle of orange drink, which contains 0.75*l*, costs 2 dollars. What is the price of 1*l* ?

P2. A bottle of orange drink, which contains 2*l*, costs 6 dollars. What is the price of 1*l* ?

If one asks to solve only P1, hiding P2 from sight, one always notes, among those present, quite a long moment of embarrassment. Then doing P2 shortly after, many will be ready sincerely to admit that, while the second problem is solved immediately by the division 6:2, solving the first with the *analogous* division 2:0.75 creates strong embarrassment.

Let's look at the comment made by Fischbein (1985) himself: «As a consequence, one can suppose that it is precisely the numbers and the relationships between them which block or facilitate the recognition of the division operations as resolutive procedure. Each arithmetic operation also possesses, above and beyond its *formal meaning*, one or more *intuitive meanings*. The two levels can coincide or not».
[A much deeper treatment of the analysis of these and many other cases is found in (D'Amore, 1999, 2003). For this reason, I pass over them here].

I have often tried asking the teachers and the more mature students how they would have solved P1. Some confessed to having considered 0.75 as ¾ and having therefore proceeded into the field of fractions (not always working in an irreproachable way). Others, instead, admitted to having solved P1 with the proportion 0.75:2=1:*x* and having then applied the known properties for solving it (with success). Now, one minds well, in the course of the resolution of this linear equation in the unknown x, there is a moment in which one *must* do 2:0.75, which is apparently the same operation that, if carried out directly on the data of the problem, would have solved P1 in a wink. However, it is not the same thing. If it is true, as it is indisputably true, that there is a strong resistance in many of us to carrying out directly 2:0.75 (because of the contrast between the formal meaning and intuitive meaning of division), there is no more any embarrassment in applying the rules of proportions and to carrying out the *steps of an algorithm*, when this arrives at the final moment asking us to carry out the *apparently same* operation. Here, as we now know, a clause of the didactic contract is set off; that of *formal proxy*. In a certain sense, we do not anymore directly endeavour to perform that step; it is no longer a question of choice, of personal decision. One entrusts himself to the algorithm, to the calculation, the resolution of the problem, a sort of relieving oneself of responsibility on the part of the solver.[8]

In the course of a test of the pupils' ability in finding the roots of II degree equations, the teacher proposed, amongst others, the equation (x-1)(x-3)=0. This had never happened before. The equations had never been given in the form of product of binomials, but in their canonical form. In all of the class, the students interpreted the handout as a constraint determined by the didactic conditions which have as their aim having them produce the expected answer, independently of its modality of production. Therefore, instead of answering immediately +1 and +3, they multiplied between them the two binomials arriving at the equation in the usual canonical form, and only then supplying the two expected roots +1 e +3. [Obviously several got the calculations wrong, producing completely different roots.] This behaviour, it is useless to insist, is explained well with the didactic contract.

CONCLUSION

I would not want to give the idea that the didactic contract takes effect in the classroom only on young pupils and at low levels of school attendance. Examples at high levels abound, even at the university and even in the courses for teachers of mathematics, in initial formation and in service Fandiño Pinilla, 2005; Fandiño Pinilla, D'Amore, 2006). It has to do, therefore, with a powerful tool for analysing classroom events, one of many that have been given to us by the passionate and decades-long studies of Guy Brousseau, the indubitable pioneer in this field.

Compared with his initial ideas that evolved over time many researchers have done everything possible to find examples and to explore the concept even more deeply.

[8] On this same theme, there exists a work by Brousseau (1987); in a table on page 59, the Author analyses the following problem: Let's imagine that one pays 0.2 pounds per 0,75 litres. Brousseau affirms that the division 0.2:0.75 turns out to be more surprising than 2:0.75.

Doing so however, many Authors have wound up interpreting in many different ways the original idea (Sarrazy, 1995).

This does not limit, in my opinion, the suitability of the tool, in fact it amplifies it demonstrating, with a versatile and very powerful example, the strength of the studies which have changed our community in the last 40 years.

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Brousseau, G., 1977; between 1970 and 1973 Guy Brousseau published various articles in the notebooks of the IEM of Bordeaux which have the name: «Compte-rendu du séminaire de recherches 1971-72 et projets pour 1972-73»; but then, these publications continued until 1978. Here, I refer to one of these publications for internal use, number 18, in the version published in Barcellona in 1977.

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[9] Duly and correctly, in the bibliography various pioneering works of Guy Brousseau are cited, many of which today are extremely difficult to find, even if not expressly cited in the course of this article. I consider mine as a modest contribution to thematic historic reconstruction, a tribute to the French scholar.

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